I. Conservation of Mass

A. We will find a continuity equation for an element of fluid flowing through a fixed differential control volume that has only open control surfaces

B. Assume three dimensional flow where velocity field has components
$u = u(x,y,z,t)$, $v = v(x,y,z,t)$, $w = w(x,y,z,t)$

C. Point (x,y,z) is the center of control volume and density is defined by scalar field $\rho = \rho(x,y,z,t)$

D. Convective changes are only considered in the x-direction

E. If we apply continuity equation in the x-direction:

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \mathbf{V} \cdot d\mathbf{A} = 0$$

$$\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z + \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z - \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z = 0$$

F. Dividing by $\Delta x \Delta y \Delta z$ we get:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

G. Including convective changes in y and z direction gives us:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

H. Using gradient operator $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ and $\mathbf{V} = u\hat{i} + v\hat{j} + w\hat{k}$ we get:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

I. Two-Dimensional Steady Flow of an Ideal Fluid

1. Often the continuity equation is used for two-dimensional steady state flow of an ideal fluid. This gives us the equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \; ; \; \nabla \cdot \mathbf{V} = 0$$

2. This is same as saying volumetric dilatation rate must be zero, or volume rate of change of fluid element must be zero because density is constant

J. Cylindrical Coordinates

1. Continuity equation in terms of cylindrical coordinates r, θ, z is:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

2. For incompressible fluid, steady flow, in two dimensions:

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$$

II. The Navier-Stokes Equations

A. Real fluids are viscous so we should include viscous forces in our equations to describe flow

B. We must write our equation in terms of velocity components by relating stress components to the viscosity of the fluid and the velocity gradients

C. For these equations, we obtain:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

D. Terms on left represent "ma" while terms on right represent ΣF caused by weight, pressure, and viscosity

E. The equations above are known as Navier-Stokes Equations and apply uniform, nonuniform, steady, or nonsteady flow of an incompressible Newtonian fluid

F. Together with continuity equation, the four equations provide a means of obtaining the velocity components u, v, w and the pressure p within the flow